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FACTORIZATION METHOD AND BELAFHAL'S TRIPLE SERIES ARISING IN SCATTERING PROBLEMS

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Abstract

Triple series I_1 and I_2 arising in WKB solution of a light scattering problem are analyzed in detail with the help of the operator factorization method allowing any hypergeometric series to be expressed, in a universal and flexible manner, through simpler series. We first connect the I_1 and I_2 with triple hypergeometric series F_a and F_b. Then we set different factorized forms for F_a and F_b and establish Kummer - type transformations for these series. Further we reduce these triple series to double series and show that linear Kummer-type and quadratic Bessel-type transformations are applicable to the double series. Finally we give an example of how the basic properties of F_a and F_b can be used for finding more specific relationships for these functions. Indications are given of where essentials of the new method can be found.

Keywords: Light scattering; WKB solution; Hypergeometric series; Factorization method; Transformations and reductions of multiple series.

1. Introduction

Scattering data present fundamental interest in numerous natural and technological phenomena, radioastronomy observations, etc. Scattering problems have been the subject of intense investigation and research in course of last decades [1,2] including recent years [3-13]. The complexity of the general scattering theory for non-spherical particles makes it worth examination the possibility of applying the WKB approximation to modeling the scattering of light by spheroidal objects.

Recently, Shepelevich et al [11] and Belafhal et al [12] performed a study of the evolution of the extrema in the light scattering indicatrix of a homogeneous spheroid by using the WKB approximation. In particular, the study of the dependence of the extrema positions upon orientation of absorbing and nonabsorbing orientated spheroidal particles [13] allowed the form factor and the scattering amplitude to be presented in the form of the so-called Belafhal's triple series I_1 and I_2 (see Eqs. (1) and (2) below).

On the other hand, the diffraction and the scattering patterns resulting from the interaction of the incident light with objects which have a depth, as it was introduced by Belafhal et al [13,14], can be also connected with the series I_1 and I_2. For example, it was shown in Ref. [13] that the characteristics of the intensity in the far-field region for a diffractive hemispherical aperture illuminated by a plane wave can be described by using a particular case of Belafhal's triple series.

Consequently, for mathematical advancement of the research exposed in previous papers we are forced to consider the Belafhal's triple series because they appear in many formulas of the diffraction and the scattering theories. As these series are essentially hypergeometric functions we firstly give a comment on traditional methods of analyzing these functions (see Sec. 2) to clarify the reasons making us to give preference to a new factorization method. In Sec. 3 we recast the series I_1 and I_2 into explicitly hypergeometric form. In Sec. 4 we use the operator factorization method to represent the series I_1 and I_2 through simpler hypergeometric series. In Sec. 5, by applying the generalized Kummer transformation, we derive different expressions for the hypergeometric functions appearing as substitutes for I_1 and I_2. Reduction of the triple series to double series is given in Sec. 6 and, finally, in Sec. 7 we provide an application of the Kummer transformation to the double series. On the whole we give a general mathematical basis for further investigation of any specific property of the series I_1 and I_2 which may present interest for physical applications.

2. Remark on the standard analytical methods

"Like any part of mathematics which is very important there are many ways to look at hypergeometric functions" [15]. So many that the user of these functions often finds himself in an unsettling position when he runs out of standard functions and known results presented in handbooks and reference manuals. It is just the case with the triple series I_1 and I_2. An attempt to obtain a new relation for a new function on one's own often turns out to be a disappointing experience because the value of the numerous traditional methods being used in the literature is greatly diminished by their cumbersome, not easily comprehensible structure (recollect Lie groups and symmetric spaces mentioned by Askey [15] as fashionable and powerful methods for investigation of hypergeometric series). Moreover, as a rule, each separate method has a limited domain of applicability. In short, each class of formulas and sometimes even a class of functions should be considered in its own right. One cannot expect that a common user of hypergeometric functions and even a connoisseur of the domain would exhibit such an extraordinary universality of mathematical thinking. Therefore an attempt to employ the standard analytical methods for analyzing the series I_1 and I_2 would have encountered with severe mathematical difficulties.

With the advent of the operator factorization method [16] the situation had changed substantially. Now a researcher may have in his disposal a universal and simple calculation technique based on a limited number of factorization formulas and auxiliary identities.

That's why we use, for derivation of basic properties of the triple series, the new operator factorization method. The main advantage of this method over more common approaches is that it allows a hypergeometric series to be represented, in a universal and flexible manner, through simpler series. Using either the known properties of the simpler series or, more directly, the formulas which have been already obtained with the help of the factorization method, all basic properties of the initial series can be re-established in a straightforward manner. One of the aims of this paper is just to substantiate these statements by concrete examples.

3. Recasting of the series I_1 and I_1 into hypergeometric form

The two series confronted by Belafhal in a light scattering problem have the form [12, 13]

$$I_1 = \sum_{n=0}^{\infty} \frac{\left(-v^2/4\right)^n}{n!} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} B(n+1, \frac{k+2}{2}) \times F_2^1 \begin{bmatrix} (k+2)/2 & ; & -a^2/4 \\ 1/2, \; n+(k+4)/2 & & \end{bmatrix}, \tag{1}$$

$$I_2 = \sum_{n=0}^{\infty} \frac{\left(-v^2/4\right)^n}{n!} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} B(n+1, \frac{k+3}{2}) \times F_2^1 \begin{bmatrix} (k+3)/2 & ; & -a^2/4 \\ 1/2, \; n+(k+5)/2 & & \end{bmatrix}, \tag{2}$$

where $B(\lambda, k)$ is beta function defined by $B(\lambda,k) = \Gamma(\lambda)\Gamma(k)/\Gamma(\lambda+k)$,

and F_2^1 is a particular case of generalized hypergeometric series (see [17], vol. 1)

$$F_q^p \begin{bmatrix} a_1,\dots,a_p; x \\ b_1,\dots,b_q \end{bmatrix} = \sum_{i=0}^{\infty} \frac{(a_1,i)\dots(a_p,i)\; x^i}{(b_1,i)\dots(b_q,i)\; i!}, \tag{3}$$

where (a,i) is *simple* Pochhammer symbol

$$(a,i) = a(a+1)\dots(a+i-1) = \Gamma(a+i)/\Gamma(a).$$

A multiple hypergeometric series ${}^N F(x_1,\dots,x_N)$ is usually defined as a series of power functions $x_1^{i_1} \cdots x_N^{i_N} / [i_1! \cdots i_N!]$ with coefficients having the form of ratios of products of *compound* Pochhammer symbols

$$(\; a, m_1 \; i_1 + m_N \; i_N\;), \tag{4}$$

where $i_1,\dots,i_N$ are summation variables (for each $n = 1, \dots, N$, $0 \le i_n \le \infty$) and the *spectral numbers* $m_1,\dots,m_N$ are assumed to be *arbitrary integers* (see, for example, Refs. [18] and [19]). The elementary list of parameters corresponding to "elementary coefficient" Eq. (4) will be written in the list of parameters of ${}^N F$ as $\langle a \mid m_1,\dots,m_N \rangle$. As some of the spectral numbers implicitly occurring in Eqs. (1) and (2) (as coefficients of the summation variable k) assume half integer values our first step should lie in an appropriate change of the variable k that would permit us to get rid of half integer spectral numbers.

Previously to the substitution we combine Eqs. (1) and (2) by introducing

$$T_\delta = I_{1+\delta}\ , \quad \delta = 0,1\ , \tag{5}$$

where

$$T_\delta = \sum_{n=0}^{\infty}\left(-\frac{\nu^2}{2}\right)^n \sum_{k=0}^{\infty}\frac{\beta^n}{k!}$$
$$\times\sum_{m=0}^{\infty}\frac{\Gamma\left(1+\frac{\delta}{2}+\frac{k}{2}+m\right)\left(-a^2/4\right)^m}{\left(\frac{1}{2}+\delta,m\right)\Gamma\left(2+\frac{\delta}{2}+n+\frac{k}{2}+m\right)m!}\ . \tag{6}$$

We then break T_δ into two parts corresponding to $k = 2q+\delta$ and $k = 2q+1-\delta$, respectively, where $q = 0,1,2,....$. Transforming the resultant two series to hypergeometric form and using notation of Ref. [19] we have

$$T_\delta = \frac{\beta^\delta}{1+\delta}\,{}^3F_a + \frac{2}{3}\beta^{1-\delta}\,{}^3F_b\ , \quad \delta=0,1 \tag{7}$$

$${}^3F_a = F\begin{bmatrix}\langle 1+\delta\,|\,011\rangle : 1; & * & ; & * & ;-x,y,-z\\ \langle 2+\delta\,|\,111\rangle : *; & \frac{1}{2}+\delta; & \frac{1}{2}+\delta & & \end{bmatrix} \tag{8}$$

$${}^3F_b = {}^3F\begin{bmatrix}\langle 3/2\,|\,011\rangle : 1; & * & ; & * & ;-x,y,z\\ \langle 5/2\,|\,111\rangle : *; & \frac{3}{2}-\delta; & \frac{1}{2}+\delta & & \end{bmatrix}. \tag{9}$$

$$x \equiv \nu^2/4, \quad y \equiv \beta^2/4, \quad z \equiv a^2/4\ .$$

Both hypergeometric series ${}^3F_a$ and ${}^3F_b$ relate to the following general type

$${}^3F_g = {}^3F\begin{bmatrix}\langle a\,|\,011\rangle : b\ ;*\ ;*\ ;\xi,\ \eta,\ \zeta\\ \langle c\,|\,111\rangle : *\ ;d\ ;e\end{bmatrix}. \tag{10}$$

Such a function does not relate to a standard type of hypergeometric series. We cannot hope to find its properties in manuals and reference books. We do not think that some web sites can be found to provide us with necessary information. The existent computer algebra systems are also still a very long way from being able to help us in analyzing the functions like ${}^3F_g$. Thus we are confronted with the necessity to choose a method thatwould allow us to investigate the functions ${}^3F_a$ and ${}^3F_b$ on our own. This gives us a typical example showing that a researcher may readily find himself in a situation where he is compelled to deal with the subjects and methods of investigation deviating widely from his initial point of interest. Of course the method to be chosen should be simple, direct and encompass as many objects of conceivable interest as possible.

4. Operator factorization method (see Refs. [16] and [18-29])

This method satisfies all the above criterions. The main advantage of the method is that it allows any complicated hypergeometric series to be expressed through simpler series with the help of "Ω - multiplication" operation. We call a function $w(x_1,...,x_N)$ the Ω-product of $u(x_1,...,x_N)$ and $v(x_1,...,x_N)$ if

$$w = w(x_1,...,x_N) = u\left(\frac{d}{ds_1},...,\frac{d}{ds_N}\right)$$
$$\times\ v(x_1s_1,...,x_Ns_N)\Big|_{s_1=...=s_N=0}, \tag{11}$$

or $w(x) = u(d/ds)v(xs)\big|_{s=0}$ in case of one variable. In both cases we write $w = u * v$. The most useful factorization formula is [18,20,21]

$${}^NF\left[\langle a\,|\,m_1,...,m_N\rangle, L\ ; x_1,...,x_N\right]$$
$$\equiv \sum_{i_1=0}^{\infty}...\sum_{i_N=0}^{\infty}(a, m_1 i_1+...+m_N i_N)$$
$$\times L(i_1,...,i_N)\ x_1^{i_1}...\ x_N^{i_N}\ /\ [i_1!\ ...\ i_N!]$$
$$= F\left[a;\frac{d}{ds}\right]\,{}^NF\left[L; x_1 s^{m_1},...,x_N s^{m_N}\right]\Big|_{s=0}, \tag{12}$$

where $L(i_1,...,i_N)$ is an arbitrary coefficient. If we have several spectrally equivalent (having identical sets of spectral numbers) glueing (having more than one non-zero spectral numbers) parameters, we combine these in a double set $\mathbf{d}$

$$\left\langle \begin{matrix}a_1,...,a_p\\ b_1,...,b_q\end{matrix}\,\middle|\, m_1,...,m_N\right\rangle \equiv \langle \mathbf{d}\,|\,m_1,...,m_N\rangle.$$

The corresponding factorization formula is fairly analogous to Eq. (12). We have

$${}^NF\left[\langle \mathbf{d}\,|\,m_1\ ,...,m_N\rangle, L\ ; x_1,...,x_N\right]$$
$$= F_q^p\left[\mathbf{d};\frac{d}{ds}\right]\ {}^NF\left[L; x_1 s^{m_1},...,x_N s^{m_N}\right]\Big|_{s=0}$$

where F_q^p is the series defined in Eq. (3). Coming back to our initial problem we see that the series ${}^3F_g$ in Eq. (10) has two spectrally non-equivalent glueing parameters $\langle a\,|\,011\rangle$ and $\langle c\,|\,111\rangle$. Thus we cannot get by with only one factorization formula. By successive two-fold application of Eq. (12) we have

$${}^3F_g = F_1^0\begin{bmatrix}*; d/ds\\ c\end{bmatrix}$$

$$\times\ F\begin{bmatrix} \langle a|011\rangle : b\ ;\ *\ ;*\ ;\ \xi s,\ \eta s,\ \zeta s \\ *\ \ :*\ ;d\ ;e \end{bmatrix}\Big|_{s=0}$$

$$= F_1^0\begin{bmatrix} *;\ d/ds \\ c \end{bmatrix} F_0^1\begin{bmatrix} *;\ d/dt \\ a \end{bmatrix} F_0^1\begin{bmatrix} b\ ;\ \xi\ s \\ * \end{bmatrix}$$

$$\times\ F_1^0\begin{bmatrix} *\ ;\ \eta\ s\ t \\ d \end{bmatrix} F_1^0\begin{bmatrix} *\ ;\ \zeta s\ t \\ e \end{bmatrix}\Big|_{s=t=0} . \qquad (13)$$

The complete factorization of the ${}^3F_g$ is due to the fact that in absence of glueing parameters a multiple series brakes up into a product of simple series corresponding to individual parameters.

The advantage of the operator factorization method is clearly demonstrated by Eq. (13). Instead of the complicated analytical structure defined by Eq. (10) we have 5 very simple series in Eq. (13); three series are of Bessel type F_1^0 and two series are of binomial type F_0^1.

Looking at the initial series Eqs. (8) and (9) we feel that denominator parameters 1/2 and 3/2 may well be linked with "individual" series $F_1^0\left[*//1/2;u\right]$ and $F_1^0\left[*//3/2;u\right]$ connected with exponential function by

$$F_1^0\begin{bmatrix} *\ ;\ u \\ 1/2 \end{bmatrix} = \frac{1}{2}\left(e^{\,2\sqrt{u}} + e^{-2\sqrt{u}}\right), \qquad (14)$$

$$F_1^0\begin{bmatrix} *\ ;\ u \\ 3/2 \end{bmatrix} = \frac{1}{4\sqrt{u}}\ \left(e^{2\sqrt{u}} - e^{-2\sqrt{u}}\right) \qquad (15)$$

and

$$e^{\,u} = F_1^0\begin{bmatrix} *\ ;u^2/4 \\ 1/2 \end{bmatrix} + u\ F_1^0\begin{bmatrix} *\ ;u^2/4 \\ 2/3 \end{bmatrix}, \qquad (16)$$

$$e^{-u} = F_1^0\begin{bmatrix} *\ ;u^2/4 \\ 1/2 \end{bmatrix} - u\ F_1^0\begin{bmatrix} *\ ;u^2/4 \\ 3/2 \end{bmatrix}. \qquad (17)$$

However it would be difficult to justify this natural guess using traditional "classical" methods. On the contrary, Eq. (13) allows our hypothesis to be substantiated in a very easy and "constructive" way.To illustrate the flexibility of the factorization approach we interchange F_0^1 (d/dt) and F_0^1 (ξs) in Eq. (13) and then apply Eq. (12) to the functions containing t and d/dt. The result is

$${}^3F_g = F_1^0\begin{bmatrix} *\ ;d/ds \\ c \end{bmatrix} F_0^1\begin{bmatrix} b\ ;\ \xi\ s \\ * \end{bmatrix}$$

$$\times\ {}^2F\begin{bmatrix} a:*;*;\eta\ s,\ \zeta\ s \\ *:d\ ;e \end{bmatrix}\Big|_{s=0}, \qquad (18)$$

$${}^2F\begin{bmatrix} a:*;*;x,y \\ *:d\ ;e \end{bmatrix} \equiv \Psi_2\left[a,d,e;\ x,y\right], \qquad (19)$$

where Ψ_2 is the standard Humbert function ([17], vol. 1).

Performing such quick and useful tricks with versatile types of standard and non-standard functions is beyond the capabilities of any other existing method.

5. Generalized Kummer transformation of ${}^3F_a$ and ${}^3F_b$

We shall not use the formulas (13), (18) and (19) for the study of functions ${}^3F_a$ and ${}^3F_b$. A close inspection shows that there exists more convenient approach to examination of these functions. We first remind the definition of the partial type of a hypergeometric series. Suppose that all variables of the series ${}^N F\left(x_1,...,x_N\right)$ except of a chosen variable x_n are put equal to zero. The resultant simple series can be brought to the form of the generalized hypergeometric series (3). The set $\left[p_n//q_n\right]$ indicating the numbers of simple numerator and denominator parameters is called the *partial type* of ${}^N F$ with respect to x_n [18-21].

The most interesting cases are connected with the Bessel type $\left[0//1\right]$, the Kummer type $\left[1//1\right]$ and the Gauss type $\left[2//1\right]$. The reason is that just these types are most typical for major part of applied problems.

An interesting feature of any series ${}^N F$ having one of these three partial types is that such a series satisfies a transformation analogous to the corresponding transformation of a series in one variable. Explicit general formulas have been obtained for each of the three types [19,20,22].

The "representative" series ${}^N F$ in Eq.(10) has the partial types $\left[1//1\right]$, $\left[1//2\right]$ and $\left[1//2\right]$ with respect to ξ,η and ζ. Therefore only Kummer-type transformation out of the three special transformations can be applied to the ${}^3F_g$.

The most general Kummer-type transformation is [20-23]

$$F\begin{bmatrix} \langle \nu_1|1,\mathbf{m}_1\rangle,\ L^*;\ x_0,\ \mathbf{x} \\ \langle \nu_0|1,\mathbf{m}_0\rangle \end{bmatrix} = \exp(x_0)$$

$$\times F\begin{bmatrix} \langle \nu_{0\bar{1}}|1,\mathbf{m}_{0\bar{1}}\rangle\ \langle \nu_1|0,\mathbf{m}_1\rangle,\ L^*;-x_0,\mathbf{x} \\ \langle \nu_0|1,\mathbf{m}_0\rangle\ \langle \nu_{0\bar{1}}|0,\mathbf{m}_{0\bar{1}}\rangle \end{bmatrix}. \qquad (20)$$

In Eq. (20) we put for brevity $\nu_{0\bar{1}} = \nu_0 - \nu_1$, $\mathbf{m}_{0\bar{1}} = m_0 - m_1$, where $\mathbf{m}_1$ and $\mathbf{m}_0$ are N-component sets of integer spectral numbers and L^* symbolizes coefficient $L(i_1,...,i_N)$ independent of summation index i_0. The "vector" $\mathbf{x}=[x_1,...,x_N]$ is an N-component set of arguments.

Converting Eq. (10) into canonical form [20-22,24]

$$^3F_g = {}^3F\begin{bmatrix}\langle b|100\rangle,\ L^*;\ \xi,\ \eta,\ \zeta\\ \langle\nu_0|111\rangle\end{bmatrix}, \qquad (21)$$

$$L^* \equiv \begin{bmatrix}\langle a|011\rangle\ :\ *\ ;\ *\ ;\ *\ ;\\ :\ *\ ;\ d\ ;\ e\end{bmatrix},$$

and applying the general transformation (20) to Eq. (21) we obtain

$$^3F_g = \exp(\xi) \times F\begin{bmatrix}\langle c-b|111\rangle\ \langle a|011\rangle\ :*;*;*;-\xi,\eta,\zeta\\ \langle c|111\rangle\ \langle c-b|011\rangle:*;d;e\end{bmatrix}. \qquad (22)$$

In contrast to the case of the ^3F_g parameters *a, b* and *c* in Eqs. (8) and (9) obey the identity *a=c - b* which permits identical numerator and denominator parameters $\langle a|011\rangle$ and $\langle c-b|011\rangle$ to be cancelled out. Thus we have

$$^3F_a = \exp(x) \times {}^3F_\alpha\begin{bmatrix}\langle 1+\delta|111\rangle:*;\ *\ ;\ *\ ;x,y,-z\\ \langle 2+\delta|111\rangle:*;\frac{1}{2}+\delta;\ \frac{1}{2}+\delta\end{bmatrix}, \qquad (23)$$

$$^3F_b = \exp(x) \times {}^3F_\beta\begin{bmatrix}\langle 3/2|111\rangle:*;\ *\ ;\ *\ ;x,y,-z\\ \langle 5/2|111\rangle:*;\frac{3}{2}-\delta;\ \frac{1}{2}+\delta\end{bmatrix}. \qquad (24)$$

Unlike ^3F_a and ^3F_b we have two spectrally equivalent glueing parameters in $^3F_\alpha$ and $^3F_\beta$ which allow us, contrary to Eq. (13), to manage with only one differentiation in the factorization formulas:

$$^3F_\alpha = F_1^1\begin{bmatrix}1+\delta\ ;\ \frac{d}{d\xi}\\ 2+\delta\end{bmatrix}\exp(x\xi) \times F\begin{bmatrix}*\ ;\ y\xi\\ 1/2+\delta\end{bmatrix}F\begin{bmatrix}*\ ;-z\xi\\ 1/2+\delta\end{bmatrix}\Bigg|_{\xi=0}, \qquad (25)$$

$$^3F_\beta = F_1^1\begin{bmatrix}3/2\ ;\ \frac{d}{d\xi}\\ 5/2\end{bmatrix}\exp(x\xi) \times F\begin{bmatrix}*\ ;\ y\xi\\ 3/2-\delta\end{bmatrix}F\begin{bmatrix}*\ ;-z\xi\\ 1/2+\delta\end{bmatrix}\Bigg|_{\xi=0}. \qquad (26)$$

6. Reduction of triple series to double series

Using Eqs. (14)-(17) we express the products of $F_1^0[*//1/2;u]$ and $F_1^0[*//3/2;u]$ occurring in Eqs. (25) and (26) as

$$F\begin{bmatrix}*\ ;\ u\\ 1/2\end{bmatrix}F\begin{bmatrix}*\ ;\ v\\ 1/2\end{bmatrix} = \frac{1}{2}F\begin{bmatrix}*\ ;W_+\\ 1/2\end{bmatrix}+\frac{1}{2}F\begin{bmatrix}*\ ;W_-\\ 1/2\end{bmatrix}, \qquad (27)$$

$$W_+ = \left(\sqrt{u}+\sqrt{v}\right)^2,\ W_- = \left(\sqrt{u}-\sqrt{v}\right)^2,$$

$$F\begin{bmatrix}*\ ;\ u\\ 3/2\end{bmatrix}F\begin{bmatrix}*\ ;\ v\\ 1/2\end{bmatrix} = U_+F\begin{bmatrix}*\ ;W_+\\ 3/2\end{bmatrix}+U_-F\begin{bmatrix}*\ ;W_-\\ 3/2\end{bmatrix}, \qquad (28)$$

$$U_+ = \left(\sqrt{u}+\sqrt{v}\right)/2\sqrt{u},\ U_- = \left(\sqrt{u}-\sqrt{v}\right)/2\sqrt{u}$$

$$F\begin{bmatrix}*\ ;\ u\\ 1/2\end{bmatrix}F\begin{bmatrix}*\ ;\ v\\ 3/2\end{bmatrix} = V_+F\begin{bmatrix}*\ ;W_+\\ 3/2\end{bmatrix}-V_-F\begin{bmatrix}*\ ;W_-\\ 3/2\end{bmatrix}, \qquad (29)$$

$$V_+ = \left(\sqrt{u}+\sqrt{v}\right)/2\sqrt{u},\ V_- = \left(\sqrt{u}-\sqrt{v}\right)/2\sqrt{v}$$

$$F\begin{bmatrix}*\ ;\ u\\ 3/2\end{bmatrix}F\begin{bmatrix}*\ ;\ v\\ 1/2\end{bmatrix} = \frac{1}{8\sqrt{uv}}F\begin{bmatrix}*\ ;W_+\\ 1/2\end{bmatrix}-\frac{1}{8\sqrt{uv}}F\begin{bmatrix}*\ ;W_-\\ 1/2\end{bmatrix}. \qquad (30)$$

If we had used Eq. (30) directly for transforming the double product $F_1^0(y\xi)\ F_1^0(-z\xi)$ with $\delta=1$ in Eq. (25) we would have confronted with singularity ξ^{-1} which turns out a formal obstacle for application of the "glueing operator" $F_1^1[1+\delta//2+\delta;d/d\xi]$ at $\xi=0$. To get rid of the singularity we use the elementary transformation

$$F[\mathbf{d};\ x] = 1+(\mathbf{d},1)\ x\ F\begin{bmatrix}\mathbf{d}+1,1\ ;\ x\\ 2\end{bmatrix},$$

for the both series $F_1^0[*//1/2]$ in the right hand part of Eq. (30). The result (not containing the above mentioned singularity) is

$$F\begin{bmatrix}*\ ;\ u\\ 3/2\end{bmatrix}F\begin{bmatrix}*\ ;\ v\\ 1/2\end{bmatrix} = R_+F\begin{bmatrix}1\ ;W_+\\ 2,\ 3/2\end{bmatrix}-R_-F\begin{bmatrix}1\ ;W_-\\ 2,\ 3/2\end{bmatrix}, \qquad (31)$$

$$R_+ = \frac{(\sqrt{u}+\sqrt{v})^2}{4\sqrt{uv}},\ R_- = \frac{(\sqrt{u}-\sqrt{v})}{4\sqrt{uv}}.$$

The combined form of Eqs. (28) and (29) can be written as

$$F\left[\begin{matrix} * & ; u \\ 1/2+\delta & \end{matrix}\right] F\left[\begin{matrix} * & ; v \\ 1/2+\delta & \end{matrix}\right]$$

$$= 2^{\delta-1} R_+^{\delta} F\left[\begin{matrix} 1 & ; W_+ \\ 1+\delta,\ 1/2+\delta & \end{matrix}\right]$$

$$+ 2^{\delta-1}(-R_-)^{\delta} F\left[\begin{matrix} 1 & ; W_- \\ 1+\delta,\ 1/2+\delta & \end{matrix}\right]. \quad (32)$$

The combined form of Eqs. (32) and (25) is

$$F\left[\begin{matrix} * & ; u \\ 3/2-\delta & \end{matrix}\right] F\left[\begin{matrix} * & ; v \\ 1/2+\delta & \end{matrix}\right]$$

$$= T_+ F\left[\begin{matrix} * ; W_+ \\ 3/2 \end{matrix}\right] + (-1)^{\delta} T_- F\left[\begin{matrix} * ; W_- \\ 3/2 \end{matrix}\right], \quad (33)$$

$$T_+ = \left[\sqrt{u}+\sqrt{v}\right] / \left[2\sqrt{u}^{1-\delta}\sqrt{v}^{\delta}\right],$$

$$T_- = \left[\sqrt{u}-\sqrt{v}\right] / \left[2\sqrt{u}^{1-\delta}\sqrt{v}^{\delta}\right].$$

Inserting Eq. (32) into Eq. (25) and using the factorization formula (12) we obtain

$${}^3F_{\alpha} = \frac{1}{2}\left\{\left(\sqrt{y}+i\sqrt{z}\right)^2 / \left[2i\sqrt{yz}\right]\right\}^{\delta}$$

$$\times\ {}^2F\left[\begin{matrix} 1+\delta:*; & 1 & ; x,\ w_+ \\ 2+\delta:*; 1+\delta, & \frac{1}{2}+\delta & \end{matrix}\right]$$

$$+\frac{1}{2}\left\{-\left(\sqrt{y}-i\sqrt{z}\right)^2 / \left[2i\sqrt{yz}\right]\right\}^{\delta}$$

$$\times {}^2F\left[\begin{matrix} 1+\delta:*; & 1 & ; x,\ w_- \\ 2+\delta:*; 1+\delta, & \frac{1}{2}+\delta & \end{matrix}\right], \quad (34)$$

$$w_+ = (\sqrt{y}+i\sqrt{z})^2,\ w_- = (\sqrt{y}-i\sqrt{z})^2.$$

Inserting Eq. (33) in Eq. (26) and using the same formula (12) we have

$${}^3F_{\beta} = \frac{1}{2}\left\{1+(-1)^{\delta}(z/y)^{1/2-\delta}\right\}$$

$$\times F\left[\begin{matrix} 3/2: * ; * ; x,\ (\sqrt{y}+i\sqrt{z})^2 \\ 5/2: * ; 3/2, \end{matrix}\right]$$

$$+\frac{1}{2}\left\{1-(-1)^{\delta}(z/y)^{1/2-\delta}\right\}$$

$$\times F\left[\begin{matrix} 3/2: * ; * ; x,\ (\sqrt{y}-i\sqrt{z})^2 \\ 5/2: * ; 3/2, \end{matrix}\right]. \quad (35)$$

The formulas (34) and (35) give us the desired reduction of triple series to two double series.

7. Kummer transformation of double series

Each of the double series in Eqs. (34) and (35) has Kummer type with respect to x. To apply the Kummer transformation to these series we have to use canonical form of the series (see Eq. (20)). To this end we only need to indicate, explicitly, the spectral numbers of glueing parameters in Eqs. (34) and (35). After making this we apply the general formula (20) to yield

$${}^2F\left[\begin{matrix} \langle 1+\delta|1,1\rangle :* ; & 1 & ; x,\ w_{\pm} \\ \langle 2+\delta|1,1\rangle :* ; & 1+\delta,\ 1/2+\delta & \end{matrix}\right]$$

$$= e^{x} F\left[\begin{matrix} * & :1 ; & 1 & ; -x, w_{\pm} \\ \langle 2+\delta|1,1\rangle :* ; & 1/2+\delta & & \end{matrix}\right], \quad (36)$$

and

$$F\left[\begin{matrix} \langle 3/2|1,1\rangle : * ; * ; x,\ w_{\pm} \\ \langle 5/2|1,1\rangle : * ; 3/2, \end{matrix}\right] = e^{x}\ {}^2F\left[\begin{matrix} * : 1 ; * ; -x,\ w_{\pm} \\ 5/2 : * ; * \end{matrix}\right]. \quad (37)$$

The functions ${}^3F_a$ and ${}^3F_b$ take the form:

$${}^3F_{\alpha} = \frac{1}{2}\left\{\left(\sqrt{y}+i\sqrt{z}\right)^2 / \left[2i\sqrt{yz}\right]\right\}^{\delta}$$

$$\times {}^2F\left[\begin{matrix} * :1; & 1 & ; -x,\ w_+ \\ 2+\delta :*; & \frac{1}{2}+\delta & \end{matrix}\right]$$

$$+\frac{1}{2}\left\{-\left(\sqrt{y}-i\sqrt{z}\right)^2 / \left[2i\sqrt{yz}\right]\right\}^{\delta}$$

$$\times {}^2F\left[\begin{matrix} * :1; & 1 & ; -x,\ w_- \\ 2+\delta :*; 1+\delta, & \frac{1}{2}+\delta & \end{matrix}\right], \quad (38)$$

$${}^3F_{\beta} = \frac{1}{2}\left\{1+(-1)^{\delta}(z/y)^{1/2-\delta}\right\}$$

$$\times {}^2F\left[\begin{matrix} * :1 ; * ; -x, (\sqrt{y}+i\sqrt{z})^2 \\ 5/2 :* ; * \end{matrix}\right]$$

$$+\frac{1}{2}\left\{1-(-1)^{\delta}(z/y)^{1/2-\delta}\right\}$$

$$\times {}^2F\left[\begin{matrix} * :1 ; * ; -x, (\sqrt{y}-i\sqrt{z})^2 \\ 5/2 :* ; * \end{matrix}\right]. \quad (39)$$

Note that functions ${}^2F$ in the right hand part of Eq. (39) are standard Humbert function Φ_3

$${}^2F\left[\begin{matrix} * : 1 ; * ; -x, w_{\pm} \\ 5/2 : * ; * \end{matrix}\right] \equiv \Phi_3\left[1,\ 5/2; -x, w_{\pm}\right]. \quad (40)$$

Yet another beneficial peculiarity of the functions is that they not only have Kummer type $[1//1]$ with respect to the first argument but also the Bessel type $[0//1]$ with respect to the second argument. Therefore we can apply to these functions a general quadratic transformation [22,23]

$$F\left[\begin{matrix} L* & ; -x_0^2,\ \mathbf{x} \\ \langle \nu|1,\mathbf{m}\rangle & \end{matrix}\right] = \exp(-2x_0)$$

$$\times F\left[\begin{matrix} \langle \nu-1/2|1,\mathbf{m}\rangle, L*; 4x_0,\ 4^m\mathbf{x} \\ \langle 2\nu-1|1,2\mathbf{m}\rangle \end{matrix}\right]. \quad (41)$$

The final result is

$$F\left[\begin{matrix} * : 1 ; * ; -x, w_{\pm}^{2} \\ 5/2 : *; * \end{matrix}\right] == \exp(-2\ w_{\pm})$$

$$\times F\left[\begin{matrix} 2 : 1 ; * ; -x,\ 4w_{\pm} \\ \langle 4|2,1\rangle : * ; * \end{matrix}\right], \qquad (42)$$

where $w_{\pm} = \sqrt{y} \pm i\sqrt{z}$ (see the line following Eq.(34)).

In conclusion we remark that the relations obtained above allow us to develop versatile means for further analysis of different functions introduced in course of our investigation. To give an example, consider the ${}^{2}F$ occurring in Eqs. (39, (40). Using self explanatory transformations we have

$$^{2}F\left[\begin{matrix} * : 1 ; * ; -x,\ w^{2} \\ 5/2 : * ; * \end{matrix}\right]$$

$$= F_{1}^{0}\left[\begin{matrix} * ; d/ds \\ c \end{matrix}\right] F_{0}^{1}\left[\begin{matrix} 1 ; -xs \\ \end{matrix}\right] e^{\,w^{2}s} \Big|_{s=0}$$

$$= \sum_{n=0}^{\infty} \frac{w^{2n}}{n!} F_{1}^{0}\left[\begin{matrix} * ; d/ds \\ 5/2 \end{matrix}\right] s^{n}$$

$$\times F_{0}^{1}\left[\begin{matrix} 1 ; -xs \\ * \end{matrix}\right] \Big|_{s=0}$$

$$= \sum_{n=0}^{\infty} \frac{w^{2n}}{n!\,(5/2,n)} F_{1}^{0}\left[\begin{matrix} * ; d/ds \\ 5/2+n \end{matrix}\right] \times F_{0}^{1}\left[\begin{matrix} 1 ; -x\,s \\ * \end{matrix}\right] \Big|_{s=0}$$

$$= \sum_{n=0}^{\infty} \frac{w^{2n}}{n!\,(5/2,n)} F_{1}^{1}\left[\begin{matrix} * ; -x \\ 5/2+n \end{matrix}\right]. \qquad (43)$$

The function $F_{1}^{1}(-x)$ in Eq. (43) is incomplete γ-function examined carefully in literature (see, for example [17], vol.2, Ch.9).

8. Concluding remarks

Operator factorization method is a very young method. Being sure of great potentialities of the method for physicists, mathematicians and chemists doing much calculational work we would like to indicate what can be found in the papers cited in the reference list and where additional information can be found. The foundations of the method with appropriate illustrative examples are given, in full detail, in Ref. [20]. An overview of the method with detailed discussion of linearization Clebsch -Gordan - type theorems and addition formulas for hypergeometric functions including a generalization of an important Koornwinder formula is given in Ref. [21]. A new theory of linear transformations of multiple hypergeometric series and application of the theory to computer analysis of Gel'fand functions are given in Refs. [24,25,19]. Universal quadratic transformations and analytical continuation formulas for such series are presented in detail in Refs. [23,26].

A detailed comparison of traditional approaches with the factorization method is illustrated in Ref. [27] by generalization of a classical relation between associated Bessel - type series. Computer programs have been developed to ensure automatic performance of linear, quadratic and analytical continuation transformations for multiple hypergeometric series [19, 23, 25, 29]. An elementary introduction to the operator factorization method is given in the recent paper [28]. Additional references can be found in the above papers.

The full (Russian) texts of papers [20,23,26] are available online on the web site of mechanical and mathematical faculty of Moscow State University http://mech.math.msu.su~fpm/rus/fpmosn.htm .

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